

# A NOTE ON MINIMAL ZERO-SUM SEQUENCES OVER $\mathbb{Z}$

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**ABSTRACT.** A *zero-sum sequence over  $\mathbb{Z}$*  is a sequence with terms in  $\mathbb{Z}$  that sum to 0. It is called *minimal* if it does not contain a proper zero-sum subsequence. Consider a minimal zero-sum sequence over  $\mathbb{Z}$  with positive terms  $a_1, \dots, a_h$  and negative terms  $b_1, \dots, b_k$ . We prove that  $h \leq \lfloor \sigma^+ / k \rfloor$  and  $k \leq \lfloor \sigma^+ / h \rfloor$ , where  $\sigma^+ = \sum_{i=1}^h a_i = -\sum_{j=1}^k b_j$ . These bounds are tight and improve upon previous results. We also show a natural partial order structure on the collection of all minimal zero-sum sequences over the set  $\{i \in \mathbb{Z} : -n \leq i \leq n\}$  for any positive integer  $n$ .

## 1. INTRODUCTION

We shall follow the notation and definitions in Gryniewicz's new monograph, and refer the reader to it for the definitions that were omitted here.

For all integers  $x$  and  $y$  with  $x \leq y$ , let  $[x, y] = \{i \in \mathbb{Z} : x \leq i \leq y\}$ . Let  $G_0$  a non-empty subset of an additive abelian group  $G$ . Let  $\mathcal{F}(G_0)$  denote the free multiplicative abelian monoid with basis  $G_0$ , and whose elements are the (unordered) sequences with terms in  $G_0$ . The identity element of  $\mathcal{F}(G_0)$ , also called *trivial sequence*, is the sequence with no terms. The operation of  $\mathcal{F}(G_0)$  is the *sequence concatenation* product that takes  $R, T \in \mathcal{F}(G_0)$  to  $S = R \cdot T \in \mathcal{F}(G_0)$ . In this case, we say that  $R$  (respectively,  $T$ ) is a *subsequence* of  $S$ . For every  $S = s_1 \cdot \dots \cdot s_t \in \mathcal{F}(G_0)$ , let

- (1) *the length* of  $S$ , denoted by  $|S|$ , be  $|S| = k$ ;
- the sum* of  $S$ , denoted by  $\sigma(S)$ , be  $\sigma(S) = s_1 + s_2 + \dots + s_t$ ;
- the average* of  $S$ , denoted by  $S_{\text{av}}$ , be  $S_{\text{av}} = \sigma(S)/|S|$ ;
- the infinite norm* of  $S$ , denoted by  $\|S\|_\infty$ , be  $\|S\|_\infty = \sup_{1 \leq i \leq t} |s_i|$ .

For any  $g \in G$  and any integer  $d \geq 0$ , we let

$$g^{[d]} = \underbrace{g \cdot \dots \cdot g}_d,$$

where  $g^{[d]}$  denotes the empty sequence if  $d = 0$ .

A *zero-sum sequence over  $G_0$*  is a sequence  $S \in \mathcal{F}(G_0)$  such that  $\sigma(S) = 0$ . Such a sequence is called *minimal* if it does not contain a proper non-trivial zero-sum subsequence.

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Then, the submonoid

$$\mathcal{B}_0 = \mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) : \sigma(S) = 0\}$$

of  $\mathcal{F}(G_0)$  is a Krull monoid (e.g., see [15]). The set  $\mathcal{A}(\mathcal{B}_0)$  of the *atoms* of  $\mathcal{B}_0$  is the set of all minimal zero-sum sequences in  $\mathcal{B}_0$ . A characterization of  $\mathcal{A}(\mathcal{B}_0)$  would shed some light on the factorization properties of  $\mathcal{B}_0$  (e.g., see [12, 13]).

Given a minimal zero-sum sequence  $S = s_1 \cdots s_t \in \mathcal{A}(\mathcal{B}_0)$ , we are interested in bounding its length in function of its terms  $s_i$  for  $i \in [1, t]$ . We are also interested in finding a natural structure for  $\mathcal{A}(\mathcal{B}_0)$  when  $G_0$  (and thus,  $\mathcal{B}_0$ ) is finite.

The study of zero-sum sequences in  $\mathcal{B}(G)$ , when  $G$  a finite cyclic group, is a very active area of research (e.g., see [2, 5, 6, 9, 18, 19, 22]) with applications to Factorization Theory (e.g., see [3, 10, 11, 12]). Similar, but less extensive, investigations have been carried out when  $G$  is an infinite cyclic group (e.g., see [4, 7, 13, 14]).

For all  $S \in \mathcal{B}(\mathbb{Z})$  with  $|S|$  finite and  $|S| > 1$ , there exist positive integers  $a_1, \dots, a_n$  and  $b_1, \dots, b_m$  with  $a_1 \leq \dots \leq a_n$  and  $b_1 \leq \dots \leq b_m$ , such that

$$(2) \quad S^+ = \prod_{i=1}^n a_i^{[x_i]}, \quad S^- = \prod_{j=1}^m (-b_j)^{[y_j]}, \quad \text{and } S = S^+ \cdot S^-,$$

where  $x_i$  and  $y_j$  are positive integers for all  $i \in [1, n]$  and  $j \in [1, m]$ .

In his work on Diophantine linear equations, Lambert [17] proved the following theorem.

**Theorem 1.1** (Lambert [17]). *Let  $S$  be a minimal zero-sum sequence over  $\mathbb{Z}$  with  $|S|$  finite and  $|S| > 1$ . If  $S$  is as in (2), then*

$$|S^+| \leq \|S^-\|_\infty = b_m \text{ and } |S^-| \leq \|S^+\|_\infty = a_n.$$

This was reformulated and reproved in the language of sequences by Baginski et al. [4]. Perhaps due to inconsistent notation across various areas, Theorem 1.1 has been independently rediscovered by Diaconis et al. [8], and Sahs et al. [21]. Currently, the best bounds for  $|S^+|$  and  $|S^-|$  are due to Henk-Weismantel [16]. They proved the following theorem for which Theorem 1.1 is a special case upon setting  $\ell = m$  and  $k = n$ .

**Theorem 1.2** (Henk-Weismantel [16]). *Let  $S$  be a minimal zero-sum sequence over  $\mathbb{Z}$  with  $|S|$  finite and  $|S| > 1$ . If  $S$  is as in (2), then*

$$(J_\ell) : \quad |S^+| \leq b_\ell - \sum_{j=1}^{\ell-1} \left\lfloor \frac{b_\ell - b_j}{a_n} \right\rfloor y_j + \sum_{j=\ell+1}^m \left\lceil \frac{b_j - b_\ell}{a_1} \right\rceil y_j \text{ for all } \ell \in [1, m],$$

$$(I_k) : \quad |S^-| \leq a_k - \sum_{i=1}^{k-1} \left\lfloor \frac{a_k - a_i}{b_m} \right\rfloor x_i + \sum_{i=k+1}^n \left\lceil \frac{a_i - a_k}{b_1} \right\rceil x_i \text{ for all } k \in [1, n].$$

In this paper, we improve on Theorem 1.2 by proving the following theorem.

**Theorem 1.3.** *Let  $S$  be a minimal zero-sum sequence over  $\mathbb{Z}$  with  $|S|$  finite and  $|S| > 1$ . If  $S$  is as in (2), then*

$$|S^+| \leq \lfloor -S_{\text{av}}^- \rfloor = \left\lfloor \frac{\sum_{j=1}^m b_j y_j}{\sum_{j=1}^m y_j} \right\rfloor \quad \text{and} \quad |S^-| \leq \lfloor S_{\text{av}}^+ \rfloor = \left\lfloor \frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n x_i} \right\rfloor.$$

The bounds in theorems 1.1–1.3 are all tight for the minimal zero-sum sequences

$$S = a^{\lfloor \frac{b}{\gcd(a,b)} \rfloor} \cdot (-b)^{\lfloor \frac{a}{\gcd(a,b)} \rfloor},$$

for all positive integers  $a$  and  $b$ . On the other hand, if we consider the minimal zero-sum sequence  $S = 3^{[1]} \cdot 4^{[2]} \cdot (-1)^{[2]} \cdot (-9)^{[1]}$ , then Theorem 1.1 yields  $|S^+| \leq 9$  and  $|S^-| \leq 4$ , Theorem 1.2 yields  $|S^+| \leq 4$  and  $|S^-| \leq 4$ , while Theorem 1.3 yields the tight bounds  $|S^+| \leq 3$  and  $|S^-| \leq 3$ .

In Section 2, we prove Theorem 1.3 by refining the method of Sahs et. al [21]. In Section 3, we define a natural partial order on the set  $\mathcal{A}(\mathcal{B}_0)$  of minimal zero-sum sequences and discuss its relevance. In Section 4, we show that the bounds in Theorem 1.3 are always sharper or equivalent to the bounds in Theorem 1.2.

## 2. PROOFS OF THEOREM 1.3

Let  $G$  be an additive abelian group, and let  $S = s_1 \cdot s_2 \dots \cdot s_t \in \mathcal{F}(G)$ . For all  $i, j \in [1, t]$  such that  $i \neq j$ , let  $S'$  be the sequence obtained by removing the terms  $s_i$  and  $s_j$  from  $S$  and inserting (anywhere) the term  $s_i + s_j$ . We call this process an  $(s_i, s_j)$ -derivation and say that  $S'$  is  $(s_i, s_j)$ -derived from  $S$ . We also say that  $S'$  is derived from  $S$  without specifying the pair  $(s_i, s_j)$ . For instance, if  $S = 2^{[3]} \cdot (-3)^{[2]}$ , then  $S' = 2^{[2]} \cdot (-3) \cdot (-1)$  is  $(2, -3)$ -derived from  $S$ , and  $S' = 4^{[1]} \cdot 2^{[1]} \cdot (-3)^{[2]}$  is  $(2, 2)$ -derived from  $S$ .

We will use the following lemma, which is a special case of Lemma 2 in Sahs et. al [21]. For the sake of completeness, we include a very short proof of it here.

**Lemma 2.1.** *Let  $G$  be an additive abelian group. Let  $S = s_1 \cdot s_2 \dots \cdot s_t$  be a minimal zero-sum sequence over  $G$ , and let  $i, j \in [1, t]$  be such that  $i \neq j$ . If  $S'$  is  $(s_i, s_j)$ -derived from  $S$ , then  $S'$  is also a minimal zero-sum sequence over  $G$ .*

*Proof.* By definition  $S'$  is a zero-sum sequence over  $G$  since  $s_i + s_j \in G$  and

$$\sigma(S') = \sigma(S) - s_i - s_j + (s_i + s_j) = \sigma(S) = 0.$$

Suppose that  $S'$  is not minimal. Then there exist nontrivial zero-sum subsequences  $R$  and  $T$  such that  $S' = R \cdot T$ , and the specific term  $s_i + s_j$  (there may be other copies of  $s_i + s_j$  in  $S'$  and  $S$ ) is a subsequence of either  $R$  or  $T$ , and not both. Thus, either  $R$  or  $T$  is a proper zero-sum subsequence of  $S$ . This would contradict the minimality of  $S$ . Thus,  $S'$  is minimal zero-sum sequence.  $\square$

We now prove our main theorem.

*Proof of Theorem 1.3.*

Let  $S$  be a minimal zero-sum sequence over  $\mathbb{Z}$  with  $|S|$  finite and  $|S| > 1$ . Then, there exist positive integers  $a_1, \dots, a_n$  and  $b_1, \dots, b_m$  with  $a_1 \leq \dots \leq a_n$  and  $b_1 \leq \dots \leq b_m$ , such that

$$S^+ = \prod_{i=1}^n a_i^{[x_i]}, \quad S^- = \prod_{j=1}^m (-b_j)^{[y_j]}, \quad \text{and} \quad S = S^+ \cdot S^-,$$

where  $x_i$  and  $y_j$  are positive integers for all  $i \in [1, n]$  and  $j \in [1, m]$ .

We shall prove by induction on  $|S| \geq 2$  that

$$(3) \quad |S^+| \leq -S_{\text{av}}^- \quad \text{and} \quad |S^-| \leq S_{\text{av}}^+.$$

If  $|S| = 2$ , then we must have  $m = n = 1$ ,  $S = a_1 \cdot (-b_1)$ , and  $a_1 - b_1 = 0$ . Since  $a_1 > 0$  and  $b_1 > 0$ , the statement (3) clearly holds. Assume that  $|S| \geq 2$  and that (3) holds for all minimal zero-sum sequence  $R$  such that  $2 \leq |R| < |S|$ .

If  $a_i = b_j$  for some  $i \in [1, n]$  and  $j \in [1, m]$ , then we must have  $S = a_i \cdot (-b_j)$ . Otherwise,  $S' = a_i \cdot (-b_j)$  would be a proper zero-sum subsequence of  $S$ , which would contradict the minimality of  $S$ . Thus, we may assume that

$$\{a_1, \dots, a_n\} \cap \{b_1, \dots, b_m\} = \emptyset.$$

Without loss of generality, we may also assume that  $a_n = \|S^+\|_\infty > \|S^-\|_\infty = b_m$ .

To prove the inductive step, we first show that  $|S^+| \leq -S_{\text{av}}^-$ . Since  $x_n > 0$ ,  $y_m > 0$ , and  $a_n - b_m > 0$ , we can use Lemma 2.1 to perform an  $(a_n, -b_m)$ -derivation from  $S$ , and obtain the minimal zero-sum sequence

$$R = (a_n - b_m)^{[1]} \cdot a_n^{[x_n-1]} \cdot \prod_{i=1}^{n-1} a_i^{[x_i]} \cdot (-b_m)^{[y_m-1]} \prod_{j=1}^m (-b_j)^{[y_j]},$$

where we omit the term  $a_n$  if  $x_n = 1$  and the term  $(-b_m)$  if  $y_m = 1$ .

Since  $|R| = |S| - 1$ , it follows from the induction hypothesis that

$$(4) \quad |R^+| = 1 + (x_n - 1) + \sum_{i=1}^{n-1} x_i = \sum_{i=1}^n x_i \leq -R_{\text{av}}^- = \frac{(y_m - 1)b_m + \sum_{j=1}^{m-1} y_j b_j}{(y_m - 1) + \sum_{j=1}^{m-1} y_j}.$$

Since  $b_m = \|S^-\|_\infty \geq \|R^-\|_\infty$ , it follows from (4) that

$$|R^+| = \sum_{i=1}^n x_i \leq \frac{b_m + (y_m - 1)b_m + \sum_{j=1}^{m-1} y_j b_j}{1 + (y_m - 1) + \sum_{j=1}^{m-1} y_j} = \frac{-\sigma(S^-)}{|S^-|} = -S_{\text{av}}^-.$$

Thus,

$$(5) \quad S^+ = \sum_{i=1}^n x_i = |R^+| \leq -S_{\text{av}}^-.$$

Next, we show that  $|S^-| \leq S_{\text{av}}^+$ . Since  $\sigma(S) = 0$ , it follows that  $\sigma(S^+) = -\sigma(S^-)$ . This observation and (5) yield

$$(6) \quad |S^+| \leq -S_{\text{av}}^- = \frac{-\sigma(S^-)}{|S^-|} = \frac{\sigma(S^+)}{|S^-|} \implies |S^-| \leq \frac{\sigma(S^+)}{|S^+|} = S_{\text{av}}^+.$$

Since  $|S^+|$  and  $|S^-|$  are integers, the theorem follows from (5) and (6) by taking the floors of  $S_{\text{av}}^+$  and  $-S_{\text{av}}^-$ .

□

**Remark 2.2.** Let  $S$  is as in (2) and suppose that there exists  $t \in [1, m]$  such that

$$(7) \quad a_n > b_t > -S_{\text{av}}^- = \frac{\sum_{j=1}^m b_j y_j}{\sum_{j=1}^m y_j}.$$

Then, the  $(a_n, -b_t)$ -derivation on  $S$  yields the minimal zero-sum sequence

$$R = (a_n - b_t)^{[1]} \cdot a_n^{[x_n-1]} \cdot \prod_{i=1}^{n-1} a_i^{[x_i]} \cdot (-b_t)^{[y_t-1]} \prod_{j=1, j \neq t}^m (-b_j)^{[y_j]}.$$

Thus, by applying Theorem 1.3 to  $R$ , we obtain

$$(8) \quad |S^+| = \sum_{i=1}^n x_i = |R^+| \leq \lfloor -R_{\text{av}}^- \rfloor.$$

Since  $-R_{\text{av}}^- < -S_{\text{av}}^-$  (by the definition of  $R$  and (7)), the bound for  $|S^+|$  in (8) is sometimes better than the bound  $|S^+| \leq \lfloor -S_{\text{av}}^- \rfloor$  given by Theorem 1.3. By symmetry, we may sometimes obtain a better bound for  $|S^-|$  in a similar manner.

### 3. THE STRUCTURE OF THE MINIMAL ZERO-SUM SEQUENCES

Let  $G_0$  be a finite subset of  $\mathbb{Z}$ . We are interested in finding a natural structure on the set  $\mathcal{A}(\mathcal{B}_0)$  of minimal zero-sum sequences in  $\mathcal{B}_0 = \mathcal{B}(G_0)$ . As mentioned in the introduction,  $\mathcal{A}(\mathcal{B}_0)$  is also the set of atoms of the Krull monoid  $\mathcal{B}_0$ . There are other interesting interpretations of  $\mathcal{A}(\mathcal{B}_0)$ . In the context of Diophantine linear equations (e.g., see [16, 17, 20]),  $\mathcal{A}(\mathcal{B}_0)$  correspond to the union of all *Hilbert bases*<sup>1</sup>, which are minimal generating sets of all the solutions. In the context integer partitions, each sequence  $S = a_1 \cdots a_p \cdot (-b_1) \cdots (-b_q) \in \mathcal{A}(\mathcal{B}_0)$  such that  $p+q \geq 3$ ,  $a_i > 0$  for  $i \in [1, p]$ , and  $b_j > 0$  for  $j \in [1, q]$ , corresponds to the *primitive partition identity*  $a_1 + \cdots + a_p = b_1 + \cdots + b_q$  (see [8, p. 1]). Primitive partition identities were studied by Diaconis et al. [8] who were motivated applications in Gröbner bases, computational statistics, and integer programming (e.g., see [23, 24]).

In the process of characterizing  $\mathcal{A}(\mathcal{B}_0)$ , we assume that  $S = s_1 \cdots s_t \in \mathcal{A}(\mathcal{B}_0)$  is equivalent to  $-S = (-s_1) \cdots (-s_t) \in \mathcal{A}(\mathcal{B}_0)$  and we only include one of them in  $\mathcal{A}(\mathcal{B}_0)$ . For any positive integer  $n$ , defined the *n-derived set*,  $\mathcal{D}_n(S)$ , of  $S = s_1 \cdots s_t \in \mathcal{B}(\mathbb{Z})$  by

$$\mathcal{D}_n(S) = \{S' : i, j \in [1, t], i \neq j, S' \text{ is } (s_i, s_j)\text{-derived, and } \|S'\|_\infty \leq n\}.$$

Given  $R, S \in \mathcal{B}(\mathbb{Z})$ , we write  $R \prec_n S$  if and only if  $R = S$  or  $R \in \mathcal{D}_n(S)$ .

The following proposition is a direct consequence of Lemma 2.1.

**Proposition 3.1.** *Let  $n$  be a positive integer,  $G_0 = [-n, n]$ , and  $\mathcal{B}_0 = \mathcal{B}(G_0)$ .*

(i) *If  $S \in \mathcal{A}(\mathcal{B}_0)$ , then  $\mathcal{D}_n(S) \subseteq \mathcal{A}(\mathcal{B}_0)$ .*

(ii)  *$\mathcal{P}_n = (\mathcal{A}(\mathcal{B}_0), \prec_n)$  is a poset.*

<sup>1</sup>This union is also known as the *Graver basis* of the corresponding *toric ideal* (e.g., see [24]).

For instance, if  $S = 2^{[3]} \cdot (-3)^{[2]}$ , then Figure 1 shows the poset  $\mathcal{P}_3$ . Note that  $S' = 2^{[3]} \cdot (-6)$  is  $(-3, -3)$ -derived from  $S$ , but  $S' \notin \mathcal{D}_3(S)$  since  $\|S'\|_\infty = 6 > 3$ .

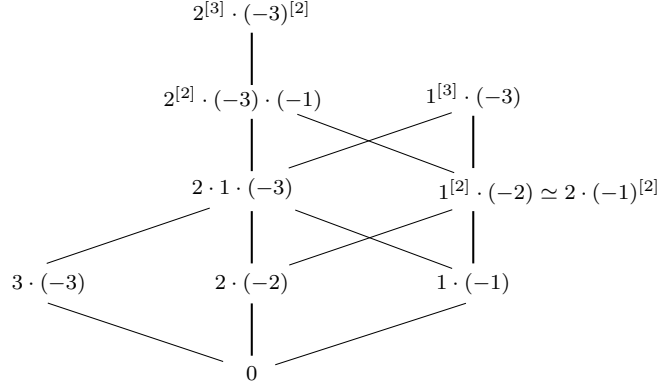


FIGURE 1. The poset  $\mathcal{P}_3$

Let  $\mathcal{M}_n$  be the set of maximal elements of the poset  $\mathcal{P}_n$  in Proposition 3.1, i.e.,  $\mathcal{M}_n$  contains all minimal sequences  $R \in \mathcal{A}(\mathcal{B}_0)$  that cannot be derived from any  $S \in \mathcal{A}(\mathcal{B}_0)$ . Then the following proposition is immediate.

**Proposition 3.2.** *Let  $n$  be a positive integer,  $G_0 = [-n, n]$ , and  $\mathcal{B}_0 = \mathcal{B}(G_0)$ . If  $\mathcal{Q}$  is a set such that  $\mathcal{M}_n \subseteq \mathcal{Q} \subseteq \mathcal{A}(\mathcal{B}_0)$ , then*

$$\mathcal{A}(\mathcal{B}_0) = \mathcal{Q} \cup \left( \bigcup_{S \in \mathcal{Q}} \mathcal{D}_n(S) \right),$$

where we assume that  $S \in \mathcal{A}(\mathcal{B}_0)$  is equivalent to  $-S \in \mathcal{A}(\mathcal{B}_0)$ .

For instance, Figure 1 shows that

$$\mathcal{M}_3 = \{2^{[3]} \cdot (-3)^{[2]}, 1^{[3]} \cdot (-3)^{[1]}\}.$$

We also verified that

$$(9) \quad \mathcal{M}_n \subseteq \left\{ a^{\lceil \frac{b}{\gcd(a,b)} \rceil} \cdot (-b)^{\lceil \frac{a}{\gcd(a,b)} \rceil} : a, b \in [1, n] \right\} \quad \text{for } n \in [1, 5].$$

However, by using the 4ti2-software package [1], we found that (9) does not hold for  $n = 6$ . In particular,

$$\begin{aligned} \mathcal{M}_6 - \left\{ a^{\lceil \frac{b}{\gcd(a,b)} \rceil} \cdot (-b)^{\lceil \frac{a}{\gcd(a,b)} \rceil} : a, b \in [1, 6] \right\} &= \left\{ 2^{[2]} \cdot 3^{[1]} \cdot 5^{[1]} \cdot (-6)^{[2]}, \right. \\ &\quad \left. 1^{[1]} \cdot 3^{[1]} \cdot 4^{[2]} \cdot (-6)^{[2]} \right\}. \end{aligned}$$

Determining  $\mathcal{M}_n$  (or a small enough superset of  $\mathcal{M}_n$ ), for all  $n > 0$ , would directly yield an algorithm for generating  $\mathcal{P}_n$ , and an approach for computing the cardinality of  $\mathcal{A}(\mathcal{B}_0)$  (e.g., by studying the Möbius function of  $\mathcal{P}_n$ ).

## 4. COMPARISON OF THE BOUNDS IN THEOREMS 1.2&amp;1.3

In this section, we show that the bounds in Theorem 1.3 are in general sharper or equivalent to the bounds in Theorem 1.2. To do this, we will show that it is enough to compare those two theorems for sequences  $S$  (where  $S$  is as in (2)) such that

$$(10) \quad a_1 \leq |S^-| = \sum_{j=1}^m y_j \leq a_n \text{ and } b_1 \leq |S^+| = \sum_{i=1}^n x_i \leq b_m.$$

First, note that it follows from Theorem 1.1 that

$$(11) \quad \sum_{j=1}^m y_j = |S^-| \leq a_n \text{ and } \sum_{i=1}^n x_i = |S^+| \leq b_m.$$

Let  $\ell \in [1, m]$ ,  $k \in [1, n]$ , and consider the upper bounds

$$(12) \quad U_{J_\ell} = b_\ell - \sum_{j=1}^{\ell-1} \left\lfloor \frac{b_\ell - b_j}{a_n} \right\rfloor y_j + \sum_{j=\ell+1}^m \left\lfloor \frac{b_\ell - b_j}{a_1} \right\rfloor y_j$$

$$(13) \quad U_{I_k} = a_k - \sum_{i=1}^{k-1} \left\lfloor \frac{a_k - a_i}{b_m} \right\rfloor x_i + \sum_{i=k+1}^n \left\lfloor \frac{a_i - a_k}{b_1} \right\rfloor x_i$$

in the inequalities  $(J_\ell)$  and  $(I_k)$  in Theorem 1.2, where  $a_1 \leq \dots \leq a_n$  and  $b_1 \leq \dots \leq b_m$ .

Without loss of generality, assume that  $b_m \geq a_n$ . Then  $\left\lfloor \frac{a_k - a_i}{b_m} \right\rfloor = 0$  for  $1 \leq i < k \leq n$ , and it follows from (13) that

$$(14) \quad U_{I_k} \geq a_k \geq a_1 \text{ for all } k \in [1, n].$$

Thus, it follows from (11), (14), and the fact that  $S_{\text{av}}^+ \geq a_1$ , that Theorem 1.2 and 1.3 can only give meaningful upper bounds for  $|S^-|$  if

$$(15) \quad a_1 \leq |S^-| = \sum_{j=1}^m y_j \leq a_n.$$

Next, it follows from the definition of  $-S_{\text{av}}^-$  in (1) that

$$(16) \quad \begin{aligned} -S_{\text{av}}^- &= \frac{-\sigma(S^-)}{|S^-|} = \frac{\sum_{j=1}^m b_j y_j}{\sum_{j=1}^m y_j} \\ &= \frac{\sum_{j=1}^m b_\ell y_j - \sum_{j=1}^{\ell-1} (b_\ell - b_j) y_j + \sum_{j=\ell+1}^m (b_j - b_\ell) y_j}{\sum_{j=1}^m y_j} \\ &= b_\ell - \frac{\sum_{j=1}^{\ell-1} (b_\ell - b_j) y_j}{\sum_{j=1}^m y_j} + \frac{\sum_{j=\ell+1}^m (b_j - b_\ell) y_j}{\sum_{j=1}^m y_j}. \end{aligned}$$

Since  $a_1 \leq \dots \leq a_n$  and  $b_1 \leq \dots \leq b_m$ , it follows from (15) and (16) that

$$\begin{aligned}
 -S_{\text{av}}^- &\leq b_\ell - \sum_{j=1}^{\ell-1} \frac{(b_\ell - b_j)y_j}{a_n} + \sum_{j=\ell+1}^m \frac{(b_j - b_\ell)y_j}{a_1} \\
 (17) \quad &\leq b_\ell - \sum_{j=1}^{\ell-1} \left\lfloor \frac{b_\ell - b_j}{a_n} \right\rfloor y_j + \sum_{j=\ell+1}^m \left\lceil \frac{b_j - b_\ell}{a_1} \right\rceil y_j = U_{J_\ell}.
 \end{aligned}$$

Thus, Theorem 1.3 and (17) yield

$$(18) \quad |S^+| \leq \lfloor -S_{\text{av}}^- \rfloor \leq -S_{\text{av}}^- \leq U_{J_\ell}$$

which implies inequality  $(J_\ell)$  in Theorem 1.2.

Moreover, it follows from (18) and the definition of  $-S_{\text{av}}^-$  that

$$(19) \quad b_1 \leq -S_{\text{av}}^- \leq U_{J_\ell}.$$

Thus, it follows from (11) and (19) that Theorem 1.2 and 1.3 can only give meaningful upper bounds for  $|S^+|$  if

$$(20) \quad b_1 \leq |S^+| = \sum_{i=1}^n x_i \leq b_m.$$

Similarly to the proof of (18), we can now use (20) to show (although we omit the details here) that Theorem 1.3 implies the inequality  $(I_k)$  in Theorem 1.2, i.e.

$$(21) \quad |S^-| \leq \lfloor S_{\text{av}}^+ \rfloor \leq S_{\text{av}}^+ \leq U_{I_k}.$$

Finally, it follows from (18) and (21) that the bounds in Theorem 1.3 are in general sharper or equivalent to the bounds in Theorem 1.2.

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